Additionstheoreme

Summe und Differenz zweier Winkel:

1. $\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$

2. $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$

3. $\tan(\alpha \pm \beta) = \dfrac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$

Doppelter und halber Winkel:

4. $\sin(2\alpha) = 2\ \sin(\alpha)\cos(\alpha)$

5. $\cos(2\alpha) = \cos(\alpha)^2 - \sin(\alpha)^2 = 2\ \cos(\alpha)^2 - 1 = 1 - 2\ \sin(\alpha)^2$

6. $\tan(2\alpha) = \dfrac{2\ \tan(\alpha)}{1 - \tan(\alpha)^2}$

7. $\sin\left(\dfrac{\alpha}{2}\right)^2 = \dfrac{1}{2}(1 - \cos(\alpha))$

8. $\cos\left(\dfrac{\alpha}{2}\right)^2 = \dfrac{1}{2}(1 + \cos(\alpha))$

9. $\tan\left(\dfrac{\alpha}{2}\right)^2 = \dfrac{1 - \cos(\alpha)}{1 + \cos(\alpha)}$

Summe in Produkt:

10. $\sin(\alpha) + \sin(\beta) = 2\ \sin\left(\dfrac{\alpha + \beta}{2}\right)\cos\left(\dfrac{\alpha - \beta}{2}\right)$

11. $\sin(\alpha) - \sin(\beta) = 2\ \sin\left(\dfrac{\alpha - \beta}{2}\right)\cos\left(\dfrac{\alpha + \beta}{2}\right)$

12. $\cos(\alpha) + \cos(\beta) = 2\ \cos\left(\dfrac{\alpha + \beta}{2}\right)\cos\left(\dfrac{\alpha - \beta}{2}\right)$

13. $\cos(\alpha) - \cos(\beta) = -2\ \sin\left(\dfrac{\alpha + \beta}{2}\right)\sin\left(\dfrac{\alpha - \beta}{2}\right)$

Produkt in Summe:

14. $2\ \sin(\alpha)\sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta)$

15. $2\ \cos(\alpha)\cos(\beta) = \cos(\alpha - \beta) + \cos(\alpha + \beta)$

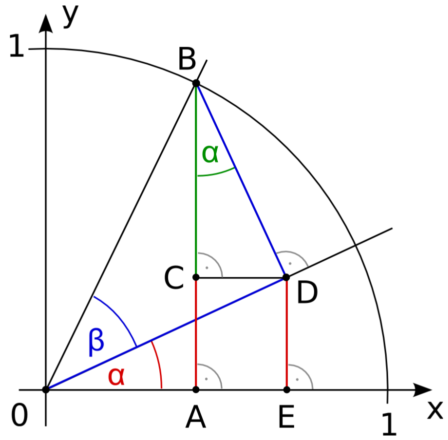
16. $2\ \sin(\alpha)\cos(\beta) = \sin(\alpha - \beta) + \sin(\alpha + \beta)$

Siehe auch Formelsammlung S 39.

Herleitung:

Wir zeigen zuerst Regel (**1**) (für $\alpha+\beta\leq\frac{\pi}{2}$). Aus dieser folgen dann alle anderen Regeln!

Mit den Bezeichnungen nebenstehender Skizze gilt:



$$\overline{0D} = \cos(\beta) \quad (I)$$

$$\overline{BD} = \sin(\beta) \quad (II)$$

$$\frac{\overline{ED}}{\overline{0D}} = \sin(\alpha) \quad \Leftrightarrow \quad \overline{ED} = \sin(\alpha)\cdot\overline{0D} \overset{(I)}{=} \sin(\alpha)\cdot\cos(\beta) \quad (III)$$

$$\frac{\overline{CB}}{\overline{BD}} = \cos(\alpha) \quad \Leftrightarrow \quad \overline{CB} = \cos(\alpha)\cdot\overline{BD} \overset{(II)}{=} \cos(\alpha)\cdot\sin(\beta) \quad (IV)$$

Insgesamt:

$$\begin{aligned}\sin(\alpha+\beta) &= \overline{AB} \\ &= \overline{AC}+\overline{CB} \\ &= \overline{ED}+\overline{CB} \\ &\overset{(III)\text{ und }(IV)}{=} \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)\end{aligned}$$

Hieraus lassen sich zwanglos die weiteren Additionstheoreme ableiten:

(1) – (3):

$$\begin{aligned}\sin(\alpha-\beta) &= \sin(\alpha)\cos(-\beta)+\cos(\alpha)\sin(-\beta) \\ &= \sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)\end{aligned}$$

$$\begin{aligned}\cos(\alpha+\beta) &= \sin\left(\frac{\pi}{2}-(\alpha+\beta)\right) = \sin\left(\left(\frac{\pi}{2}-\alpha\right)-\beta\right) = \sin\left(\frac{\pi}{2}-\alpha\right)\cos(\beta)-\cos\left(\frac{\pi}{2}-\alpha\right)\sin(\beta) \\ &= \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)\end{aligned}$$

$$\begin{aligned}\cos(\alpha-\beta) &= \cos(\alpha)\cos(-\beta)-\sin(\alpha)\sin(-\beta) \\ &= \cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)\end{aligned}$$

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)} \\ &= \frac{\dfrac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}{\dfrac{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}} = \frac{\dfrac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}+\dfrac{\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}{\dfrac{\cos(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}-\dfrac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}} \\ &= \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}\end{aligned}$$

$$\tan(\alpha-\beta) = \frac{\tan(\alpha)+\tan(-\beta)}{1-\tan(\alpha)\tan(-\beta)} = \frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$$

zu **4**) $\sin(2\alpha) = \sin(\alpha+\alpha) \overset{(1)}{=} \sin(\alpha)\cos(\alpha)+\cos(\alpha)\sin(\alpha) = 2\sin(\alpha)\cos(\alpha)$

zu **5)** $\cos(2\alpha) = \cos(\alpha+\alpha) \overset{(2)}{=} \cos(\alpha)\cos(\alpha)-\sin(\alpha)\sin(\alpha) = \underline{\cos(\alpha)^2-\sin(\alpha)^2}$

$= \cos(\alpha)^2-\left(1-\cos(\alpha)^2\right) = \underline{2\cos(\alpha)^2-1}$

$= 2\left(1-\sin(\alpha)^2\right)-1 = \underline{1-2\sin(\alpha)^2}$

zu **6)** $\tan(2\alpha) = \tan(\alpha+\alpha) \overset{(3)}{=} \dfrac{\tan(\alpha)+\tan(\alpha)}{1-\tan(\alpha)\tan(\alpha)} = \dfrac{2\tan(\alpha)}{1-\tan(\alpha)^2}$

zu **7)** $\cos(\alpha) = \cos\left(2\cdot\frac{\alpha}{2}\right) \overset{(5)}{=} 1-2\sin\left(\frac{\alpha}{2}\right)^2 \Leftrightarrow \sin\left(\frac{\alpha}{2}\right)^2 = \frac{1}{2}\left(1-\cos(\alpha)\right)$

zu **8)** $\cos(\alpha) = \cos\left(2\cdot\frac{\alpha}{2}\right) \overset{(5)}{=} 2\cos\left(\frac{\alpha}{2}\right)^2-1 \Leftrightarrow \cos\left(\frac{\alpha}{2}\right)^2 = \frac{1}{2}\left(1+\cos(\alpha)\right)$

zu **9)** $\tan\left(\frac{\alpha}{2}\right)^2 = \dfrac{\sin\left(\frac{\alpha}{2}\right)^2}{\cos\left(\frac{\alpha}{2}\right)^2} \overset{(7)\text{ und }(8)}{=} \dfrac{1-\cos(\alpha)}{1+\cos(\alpha)} = \dfrac{\frac{1}{2}\left(1-\cos(\alpha)\right)}{\frac{1}{2}\left(1+\cos(\alpha)\right)} = \dfrac{1-\cos(\alpha)}{1+\cos(\alpha)}$

zu **16)** $\sin(\alpha+\beta) \overset{(1)}{=} \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \quad (I)$

$\sin(\alpha-\beta) \overset{(1)}{=} \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \quad (II)$

$(I)+(II) \Rightarrow \sin(\alpha+\beta)+\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta)+\sin(\alpha)\cos(\beta)$

$= 2\sin(\alpha)\cos(\beta)$

zu **15)** $2\cos(\alpha)\cos(\beta) = 2\sin\left(\alpha+\frac{\pi}{2}\right)\cos(\beta) \overset{(16)}{=} \sin\left(\left(\alpha+\frac{\pi}{2}\right)+\beta\right)+\sin\left(\left(\alpha+\frac{\pi}{2}\right)-\beta\right)$

$= \sin\left((\alpha+\beta)+\frac{\pi}{2}\right)+\sin\left((\alpha-\beta)+\frac{\pi}{2}\right) = \cos(\alpha+\beta)+\cos(\alpha-\beta)$

zu **14)** $2\sin(\alpha)\sin(\beta) = 2\sin(\alpha)\cos\left(\alpha-\frac{\pi}{2}\right) \overset{(16)}{=} \sin\left(\alpha+\left(\beta-\frac{\pi}{2}\right)\right)+\sin\left(\alpha-\left(\beta-\frac{\pi}{2}\right)\right)$

$= \sin\left((\alpha+\beta)-\frac{\pi}{2}\right)+\sin\left((\alpha-\beta)+\frac{\pi}{2}\right) = -\cos(\alpha+\beta)+\cos(\alpha-\beta)$

zu **10)** $2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \overset{(16)}{=} \sin\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right)+\sin\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right) = \sin(\alpha)+\sin(\beta)$

zu **11)** $2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) \overset{(16)}{=} \sin\left(\frac{\alpha-\beta}{2}+\frac{\alpha+\beta}{2}\right)+\sin\left(\frac{\alpha-\beta}{2}-\frac{\alpha+\beta}{2}\right)$

$= \sin(\alpha)+\sin(-\beta) = \sin(\alpha)-\sin(\beta)$

zu **12)** $2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \overset{(15)}{=} \cos\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right)+\cos\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right) = \cos(\alpha)+\cos(\beta)$

zu **13)** $-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \overset{(14)}{=} -\left[\cos\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right)-\cos\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right)\right]$

$= -\left[\cos(-\beta)-\cos(\alpha)\right] = -\left[\cos(\beta)-\cos(\alpha)\right] = \cos(\alpha)-\cos(\beta)$

Bemerkung:

Mit $e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $z \in \mathbb{C}$ und $\sin(x) = \mathrm{Im}\left(e^{ix}\right)$, $\cos(x) = \mathrm{Re}\left(e^{ix}\right)$, $x \in \mathbb{R}$

gilt: $e^{ix} = \cos(x) + i \cdot \sin(x)$. Somit ist

$$\begin{aligned} \cos(x+y) + i \cdot \sin(x+y) &= e^{i(x+y)} = e^{ix} \cdot e^{iy} = \left[\cos(x) + i \cdot \sin(x)\right] \cdot \left[\cos(y) + i \cdot \sin(y)\right] \\ &= \left[\cos(x)\cos(y) - \sin(x)\sin(y)\right] + i \cdot \left[\sin(x)\cos(y) + \sin(y)\cos(x)\right] \end{aligned}$$

Vergleich der Real- und Imaginärteile ergibt die Additionstheoreme für $\sin$ und $\cos$.